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The Contribution of R. Conti to the Comparison Method in Differential Equations

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In a recent paper of A. Bacciotti and L. Pandolfi [2], the authors present a rather detailed analysis of the research work and activity of the late Professor Roberto Conti. Their analysis addresses several aspects of Conti's work and directions he has embraced in the Theory of Differential Equations and Control Theory, emphasizing his outstanding contribution brought during the entire active period (1948-2007).

The aim of this Note is to bring some complementary comments, particularly related to the ideas contained in the papers [20], [21], which in the list of publications provided in [2], correspond to the Ref. 34 and 35. These papers contain a result which lies at the ground of what is usually called *The Comparison Method* in the Theory of Differential Equations, including the ordinary case, the equations with delayed argument and other classes of generalized differential equations (for instance, equations with causal operators). The literature displays a large number of journal papers and monographs/books, in which this method is amply illustrated: [3]-[19]. The number of journal papers is very large, and we shall limit our list to [20]-[28].

In brief, Conti's approach relies on the concept of Liapunov's function and differential inequalities. In contrast with his predecessors, who have used the method with particular choices for the Liapunov's function or the differential inequality, Conti has embraced the general case, when both involved entities are of the most general type. In other words, if the differential system is

$$\dot{x}(t) = f(t, x(t)), \quad t \in \mathbb{R}+, \tag{1}$$

and $V(t,x)$ is a (general) Liapunov's function defined on $\mathbb{R}_+ \times B_r$, where $B_r = \{x;\ x \in \mathbb{R}^n,\ \|x\| < r\}$, then the inequality involved is

$$V'(t,x) \leq \omega(t, V(t,x)). \tag{2}$$

In (2), $V'(t,x)$ stands for

$$V'(t,x) = V_t(t,x) + \langle f, \operatorname{grad} V(t,x)\rangle \tag{3}$$

in case V is differentiable, while $\omega(t,y)$ represents a function with real values, defined on

$\mathbb{R}_+ \times B_r$ (or on $\mathbb{R}_+ \times \mathbb{R}^n$). From (1) and (2) there results the inequality

$$V(t, x(t; t_0, x^0)) \leq y(t; t_0, y^0), \quad t \in [t_0, T), \tag{4}$$

where $x(t; t_0, x^0)$ is the solution of (1), under condition $x(t_0) = x^0 \in \mathbb{R}^n$, $t_0 \geq 0$, while $y(t; t_0, y^0)$ is the solution of the comparison equation

$$\dot{y}(t) = \omega(t, y(t)), \tag{5}$$

such that $y(t_0) = y^0 \in \mathbb{R}$. Obviously, $T > t_0$ is the right end of the interval of existence of both solutions $x(t; t_0, x^0)$, $y(t; t_0, y^0)$. Under suitable conditions of data, from (4) one derives that $x(t; t_0, x^0)$ exists at least on the maximal interval of existence of $y(t; t_0, y^0)$, provided $V(t_0, x^0) \leq y^0$.

This is the scheme which has been used by R. Conti in the papers quoted above, in which Conti has obtained new results of a global nature, or estimates for the solutions of systems like (1).

Conti's papers [20], [21] have inaugurated a long series of publications, including monographs, textbooks and journal papers. We are providing a partial list of such publications, which are dedicated to the use of the Comparison Method in the theory of differential equations.

The above described scheme is only the beginning for a series of generalizations, including delay equations and other types of functional equations, vector Liapunov's functions and even matrix Liapunov's functions.

The method has been extended also to the infinite-dimensional case and to the dynamical systems on general metric spaces. We are not concerned with such cases, but we mention them to illustrate the proliferation of the method in the contemporary research.

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